Linear Scaling Laws in Bifurcations of Scalar Maps

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A global scaling property for bifurcation diagrams of periodic orbits of smooth scalar maps with both one and two dimensional parameter spaces is examined. It is argued that for both parameter spaces bifurcations within a periodic window of a given scalar map are well approximated by a linear transformation of the bifurcation diagram of a canonical map.

Universal metric properties of bifurcation diagrams are some of the more striking phenomena in dynamical systems. A well-known example is the observation by Feigenbaum that for many one-parameter families of scalar maps, as one follows a cascade of period doubling bifurcations in parameter space, the ratio of distances between successive period-doubling parameter values tends to approach a limiting ratio independent of the family of maps. Our work, [1] and [2], is concerned with a global scaling property for bifurcation diagrams of periodic orbits of smooth scalar maps with both one- and two-dimensional parameter spaces. Specifically, we examine "windows" of periodic behavior within chaotic regions of parameter space. In both the one-parameter and two-parameter cases we show that there is a canonical family of maps such that typically the bifurcations within a periodic window of a given scalar map are well approximated by a *linear* transformation of the bifurcation diagram of the canonical map.

The possibility of such a linear scaling law was first suggested for one-parameter families of maps by the numerical observations in [1]. More recently we have also observed this phenomenon for two-parameter families [2].

We find that a central feature of a region of periodic stability surrounded by chaotic behavior is a point in parameter space at which the map has a "superstable" orbit [3] – a periodic orbit which includes a critical point of the map. Near a superstable period n orbit, the nth iterate of the map is generally well approximated by a pure quadratic map, and the canonical bifurcation diagram in the one-parameter case is found to be that of the quadratic family $x \to \mu - x^2$. In our work, we formulate precise conditions under which we can prove a close linear correspondence between a given periodic window and the canonical diagram, and discuss the consequences of this result.

In the two-parameter case, the orbit will be superstable along a curve in parameter space. In general we expect that along lines transverse to the curve of superstability, the bifurcation diagram will resemble a one-parameter diagram to which the above scaling result applies. Indeed this should always be the case of the map has only one critical point. However, if the map has more than one critical point, then somewhere along the curve of superstability we can expect the orbit to become "doubly superstable" – to include a second critical point. Near such a point, as shown in [2], we find that the parameter region of stability for the orbit is at its thickest, and the bifurcation diagramm is not well described by the one-parameter scaling result.

Near a doubly superstable orbit of period n, the nth iterate of the map is well approximated by the composition of two quadratic maps, each of which depends linearly on the parameters. Typically each will depend on a different linear combination of the parameters, and a linear change of coordinates leads to the canonical two-parameter family $x \to (x^2 - a)^2 - b$. In our work, we state and prove [2] result for this case and discuss its ramifications.

For two-parameter families of maps [4], [5], instead of a bifurcation diagram which includes both parameter and phase space variables, one generally draws the diagram entirely in parameter space, assigning colors

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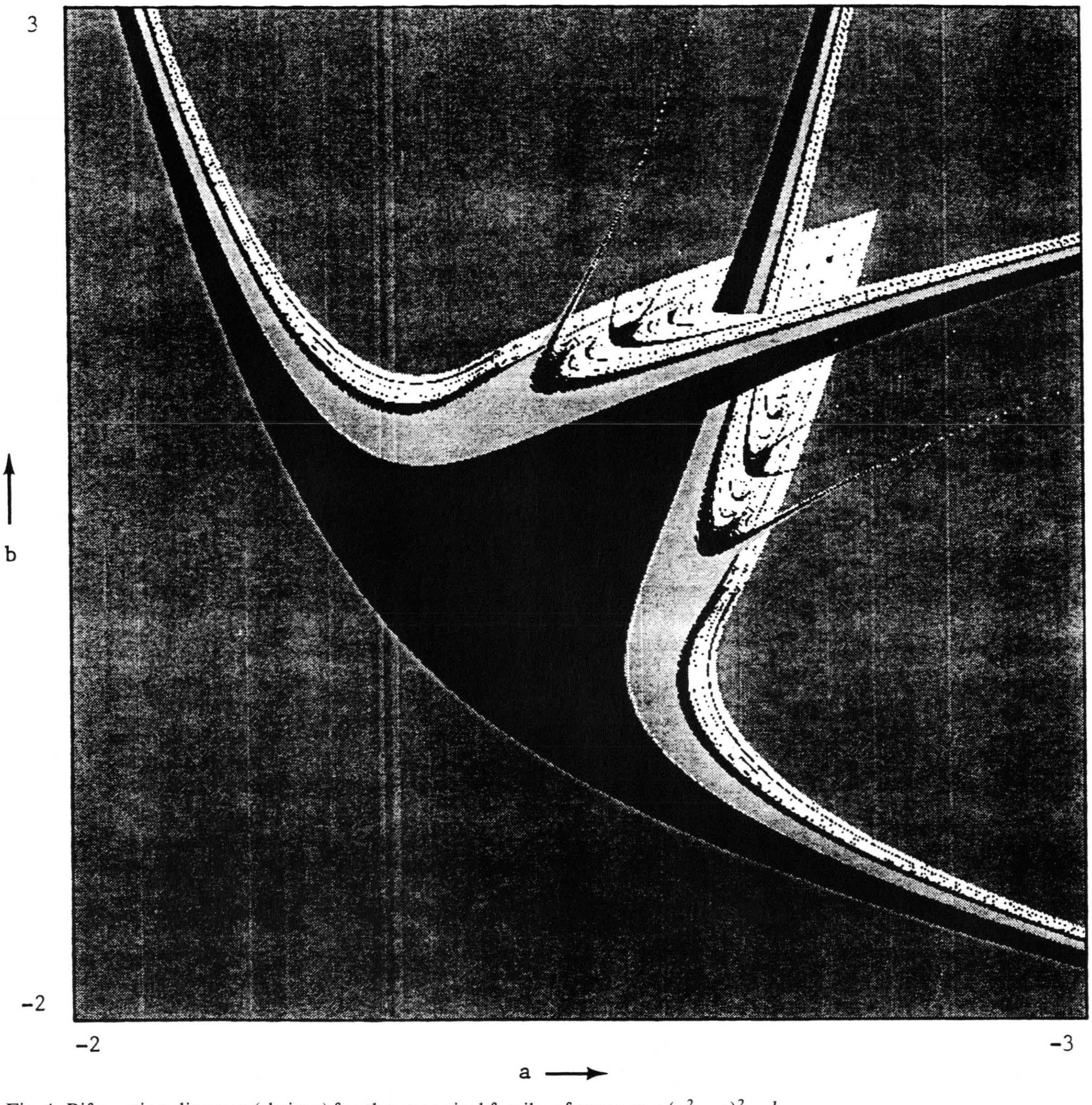

Fig. 1. Bifurcation diagram (shrimp) for the canonical family of maps $x \to (x^2 - a)^2 - b$.

or shading to regions according to the dynamics of the corresponding map. A common shape has been observed, and has been referred to as a "shrimp" [6]. This shape is seen in Fig. 1, which shows the bifurcation diagram for the canonical family of maps $x \to (x^2 - a)^2 - b$.

To understand the linear scaling, consider a window in the chaotic regime for $x_{n+1} = \mu - x_n^2$ as seen in Figure 2. Let μ_0, μ_d, and μ_c denote the values of μ at the initial saddle-node bifurcation, the first period-doubling bifurcation, and the final crisis. Then from the normalized crisis value for that window follows

$$m_c = (\mu_c - \mu_0)/(\mu_d - \mu_0). \tag{1}$$

For many of the windows, m_c is close to $\frac{9}{4}$. Furthermore, this tendency increases as the window size be-

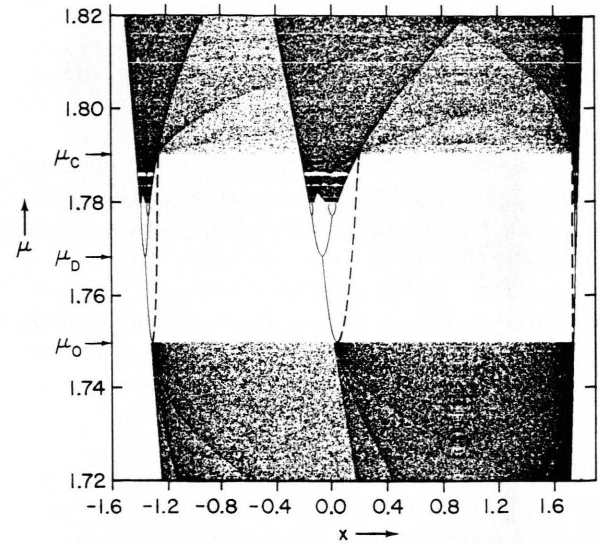

Fig. 2. Computer-generated bifurcation diagram for the map $x_{n+1} = \mu - x_n^2$ in the range $1.72 < \mu < 1.82$. A period-3 window exists in this range. The window begins with a saddle-node bifurcation (at $\mu = \mu_0 \cong 1.75$) at which stable and unstable period-3 orbits are born. The unstable orbit is indicated by the dashed line in the figure. The stable period-3 orbits goes through a period-doubling cascade and becomes chaotic as μ is increased. The end of the window occurs at the crisis, $\mu = \mu_c \cong 1.79$, when the unstable period-3 created at the original saddle-node bifurcation at $\mu = \mu_0$ (dashed line) collides with the three-piece chaotic attractor.

comes smaller. We claim that the following statement applies to any chaotic dynamical process with windows.

1. Pick any small $\varepsilon > 0$ and some interval of the system parameter which includes attracting chaotic orbits. Determine the fraction of windows of period $\leq n$ in the chosen parameter interval for which m_c deviates from $\frac{9}{4}$ by an amount than ε, $|m_c - \frac{9}{4}| < \varepsilon$. Then this fraction will approach 1 as $n \to \infty$, no matter how small ε is.

Several comments are in order concerning Statement 1:

(a) It appears that the period n does not have to be very large for many of the observed m_c to be close to $\frac{9}{4}$.

(b) The reason for our formulation of Statement 1 in terms of the *fraction* of windows is because, even for arbitrarily high-period windows there may be some few for which m_c deviates substantially from $\frac{9}{4}$.

(c) Statement 1 relates the *global* properties of a window in that m_c is a property of the map for the entire range of μ within the window.

(d) An alternative statement to Statement 1 would be that, within a period n window, the dynamics generated by the nth iterate of the map, when linearly rescaled, is typically well approximated by the canonical one-dimensional quadratic map,

$$u_{n+1} = n_n + u_n^2 - m, \tag{2}$$

where this form is such that the original saddle-node bifurcation and the first period doubling occur at $m = 0$ and $m = 1$, respectively. Thus normalizations as in (1), are automatic [e.g., for (2) the final crisis is at $m = \frac{9}{4}$].

(e) Remark (d) implies that the choice of the quantity m_c is somewhat arbitrary in that a statement analogous to Statement 1 applies to any other parameter value marking a characteristic event in the window; e.g., replace (1) by $m_3 = (\mu_3 - \mu_0)/(\mu_d - \mu_0)$, where μ_3 marks the beginning of the period-$3n$ window within the period-n window.

Concerning comment (b), an illustration is instructive. The quadratic map, $x_{n+1} = \mu - x_n^2$, has a period-3 window within its chaotic band (Fig. 2), and the value of m_c for this window is $\frac{9}{4} - 0.074\ldots$ Now consider a period-n window with n large, and assume that (2) provides a good approximation to the dynamics within this window. Clearly, within this window there is a period-$3n$ window with m_c approximately equal to $\frac{9}{4} - 0.074\ldots$ Thus, if one considers the class of windows of period $3n$ which occur as windows within period-n windows, then, for this class, m_c is *not* expected to approach $\frac{9}{4}$ as $n \to \infty$ (rather it is expected to approach $\frac{9}{4} - 0.074\ldots$). However, according to our claim in Statement 1, as n is made large, if one considers *all* orbit of period-$3n$ (not just those which occur as windows within period-n windows), then the vast majority will have m_c closely approximated by $\frac{9}{4}$. In this respect the type of universal behavior discussed here differs from other previously studied universal behavior.

Statement 1 is established rigorously [2]. Next we present an intuitive argument of its validity. Consider a period-n window with $n \gg 1$. At the final crisis for the window there will be n chaotic bands, $S_1, S_2, \ldots, S_n$, each of width $s_1, s_2, \ldots, s_n$, where we have that under the action of the map $S_1 \to S_2 \to \ldots \to S_n \to S_1$, and we choose S_1 to include the critical point, i.e., the maximum of the map function. (There must be one such interval, since otherwise there would be no folding and hence no chaos.) We assume for now that the s_j are small and that for $j \neq 1$ the location of the S_j are sufficiently far from the critical point that the map in the intervals $S_2, S_3, \ldots, S_n$ may be regarded as approximately linear. Let λ_j denote the magnitude of the slope of the map function in the middle of the interval $j (j \neq 1)$. Thus $s_{j+1} \cong \lambda_j s_j$, $n \geq j \geq 2$, and $s_2 \cong K s_1^2$. (For example, $K = \frac{1}{4}$ for $x_{n+1} = \mu - x_n^2$.) Application of these estimates to the entire cycle yields $s_1 = K \lambda^{n-1} s_1^2$, where λ is defined by $\lambda^{n-1} = \lambda_2 \lambda_3 \ldots \lambda_n$, and we call λ the *reduced Lyapunov number*. Thus

$$s_1 \sim \lambda^{-(n-1)}, \tag{3}$$

$$s_j \sim \lambda^{-2(n-1)+(j-2)}, \quad n+1 \geq j \geq 2. \tag{4}$$

Typically we expect λ to be almost constant within the window and larger than 1, reflecting the fact that the orbit, for parameter values outside the window, is chaotic. Within the assumptions above, the n-times iterated map restricted to S_1 can be regarded as the composition of one map, which is quadratic, with $n-1$ approximately linear maps. The result is an approximately quadratic map. The typical closest approach of one of the $S_j (j = 2, \ldots, n)$ to the critical point is $1/n$, which is much greater than s_j since

$$1/n \gg \lambda^{-(n-1)}, \tag{5}$$

for large n. As n increases, (5) becomes better and better satisfied, and we expect that the composed map is more and more closely approximated by a quadratic map. Furthermore, the range of μ within the window is small; in fact, as we shall show,

$$\mu_c - \mu_0 \sim \lambda^{-2(n-1)}. \tag{6}$$

Thus the variation of the λ_j with μ can be neglected, and the effect of varying μ is predominantly that of raising or lowering the level of the critical point. Hence the n-times composed map can, under linear rescaling, be put in the form of (2) and Statement 1 follows. This will be shown in more detail shortly. To see why we must formulate Statement 1 in terms of the fraction of orbits, recall our assumption in the above heuristic argument that the closest approach of $S_j (j = 2, \ldots, n)$ to the origin was $\sim 1/n$. This statement is based on the idea that the orbit for $j = 2, \ldots, n$ is, in some sense, like a chaotic orbit with Lyapunov number λ. According to this point of view, most of the period-n orbits will satisfy our assumption. However, when considering all the $\sim 2^{n-1}/n$ windows of period-n, there is always some "probability" that one of the elements S_j for $n \geq j \geq 2$ will fall too close to the critical point for the linear approximation to hold. As we look at higher n and include more orbits, we should encounter some band orbits of this type.

We now outline more formally how the rescaling yielding Eq. (2) can be obtained. Let $T(x, \mu)$ be a twice differentiable one-dimensional map with a single quadratic maximum (at $x = 0$) and a parameter μ, $x_{n+1} = T(x_n, \mu)$. Assume that T has a period-n window in $\mu_0 < \mu < \mu_c$. At $\mu = \mu_0$ there is a saddle-node bifurcation. Hence $T_x^n(\bar{x}_j, \mu_0) = 1$, where $\bar{x}_j$ are the points of a period-n orbit, $\bar{x}_{j+1} = T(\bar{x}_j, \mu_0)$ with $\bar{x}_{j+n} = \bar{x}_j$, T^n denotes the n-times composed map and $T_x^n = \partial T^n / \partial x$. Let $j = 1$ be chosen so that $\bar{x}_1$ is the closest $\bar{x}_j$ to $x = 0$. Define $\bar{x}(\mu)$ so that $\bar{x}(\mu_0) = \bar{x}_1$ and $T_x^n(\bar{x}, \mu) = 1$. [Note that for $\mu > \mu_0$, $\bar{x}(\mu)$ is not a member of a period-n orbit.] Let

$$L = T_x(\bar{x}_2, \mu_0)\, T_x(\bar{x}_3, \mu_0) \ldots T_x(\bar{x}_n, \mu_0),$$

where $T_x = \partial T / \partial x$. That is, $L = \lambda^{n-1}$ at $\mu = \mu_0$. Now introduce the rescaling $v = (x - \bar{x}) L$ and $q = (\mu - \mu_0) L^2$. Consider x_{nk} for x_{nk} in the region close to the maximum of T [i.e., $x_{nk} \sim L^{-1}$, cf. (3)], and the map $x_{n(k+1)} = T^n(x_{nk}, \mu)$. We define a map f, $v_{k+1} = f(v_k, q)$ where v_k corresponds to x_{nk}. Substituting the definitions of v and q into $x_{n(k+1)} = T^n(x_{nk}, \mu)$ and expanding for large L we obtain

$$v_{k+1} \cong v_k + \tfrac{1}{2} f_{vv}(0,0)\, v_k^2 + f_q(0,0)\, q, \tag{7}$$

where the error in (7) can be shown to be small for large $L = \lambda^{n-1}$ (i.e., large n), provided that points of the x orbit, other than the x_{nk}, do not come too close to $x = 0$, as in (5). Finally, if we set $\mu = f_{vv}(0,0)\, v/2$, $m = f_{vv}(0,0) f_q(0,0)\, q/2$, (7) becomes (2). [Note, in addition that the scaling $q = (\mu - \mu_0) L^2$ implies the estimate, (7).]

In conclusion, we have argued that typical period-n windows exhibit a global scaling structure. Numerically, we find that the scaling is already apparent at low n.

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